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Quantum Analogue of Ermakov Systems and the Phase of the Quantum Wave Function

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Based on the multidimensional Ermakov theory, a general result that relates the Schrodinger equation and the Milne equation in terms of a space invariant is established. Using this result not only the role of phase in the Wigner function approach to quantum mechanics is demonstrated but also a better explanation for the Aharonov–Bohm effect is sought in terms of a fundamental phase and the matter-field-coupling current. The existence of a similar space invariant is also emphasized for the nonlinear Schrodinger equation.

1. INTRODUCTION

Although there are many situations when quantum mechanical results reduce to the classical ones as a limiting case, relatively fewer routes and inroads from classical mechanics into the quantum domain have been discussed in the literature. In this regard, while the path integral approach in terms of the Feynman propagator has been known (Khandekar and Lawande, 1986) for a long time now, the use of Hamilton–Jacobi theory (Ghose, 1998) and the topological concept of connectedness with regard to the double-slit experiment (Varma, 1998) have been discussed recently. The aim of this paper is to provide an altogether different route, not so well known thus far, which is through the classical mechanics of time-dependent (TD) harmonic oscillator.

In Section 2, we present the quantum analogue of the classical Ermakov system with reference to the TD harmonic oscillator. Based on this, we establish a general result in Section 3 for one-dimensional Schrodinger equation (SE). Here, applications of this result are demonstrated for the Wigner function approach to quantum mechanics (QM), and they are hinted for several other physical problems. In Section 4, a generalization of this result to the three-dimensional case is carried out and the applications are further demonstrated with reference to the use of

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nonlinear Schrodinger equation (NLSE) and the studies of Aharonov–Bohm (A-B) effect.

2. QUANTUM ANALOGUE OF ERMAKOV SYSTEMS

Ermakov (1880) originally suggested a connection between the solutions of a pair of coupled differential equations. Ray and Reid in a series of papers (Ray and Reid, 1979a,b; Reid and Ray, 1979, 1982) and several others (Athorne, 1991; Eliezer and Gray, 1976; Kaushal and Korsch, 1981; Leach, 1991; Lewis and Riesenfeld, 1969; Lutzky, 1980) have exploited such a connection in the studies of TD harmonic oscillator and its possible generalizations.

It is well known that the TD harmonic oscillator described by

$$\overset{\circ\circ}{x}\ (t) + \omega^2(t)x = 0, \tag{1}$$

admits an Ermakov invariant (Athorne, 1991; Eliezer and Gray, 1976; Kaushal and Korsch, 1981; Leach, 1991; Lewis and Riesenfeld, 1969; Lutzky, 1980; Ray and Reid, 1979a,b; Reid and Ray, 1979, 1982)

$$I = c^2(x/\rho)^2 + (\overset{\circ}{x}\rho - x\overset{\circ}{\rho})^2, \tag{2}$$

where the auxiliary variable $\rho(t)$ satisfies the Milne equation

$$\overset{\circ\circ}{\rho}\ (t) + \omega^2(t)\rho = c^2/\rho^3. \tag{3}$$

Korsch and his coworkers (Korsch and Laurent, 1981; Korsch *et al.*, 1982) and also Lee (1982, 1984) noticed an interesting and striking similarity between Eq. (1) and the stationary state SE.

$$\psi''(x) + k^2(x)\psi(x) = 0, \tag{4}$$

merely by identifying x, t, ω in (1), respectively, by ψ, x, k in (4). In this new description, although Eqs. (2) and (3), respectively, take the forms

$$K = c^2(\psi/A)^2 + (\psi' A - \psi A')^2 \tag{5}$$

and

$$A''(x) + k^2(x)\ A(x) = c^2/A^3(x), \tag{6}$$

Korsch *et al.* (Korsch and Laurent, 1981; Korsch *et al.*, 1982) emphasize the connection between the solutions of (4) and (6) as

$$\psi(x) = NA(x)\ \sin\left(c\int^x A^{-2}(x')\,dx' - \delta\right), \tag{7}$$

and the new quantization rule (called Milne's quantization condition) as

$$c\int_{-\infty}^{\infty} A^{-2}(x)\,dx = (n+1)\pi, \quad n = 0, 1, 2, \ldots, \tag{8}$$

of which the WKB quantization condition turns out to be a special case. Several other applications of these results are discussed by Korsch *et al.* (Korsch and Laurent, 1981; Korsch *et al.*, 1982) and Lee (1984). Note that now K is a space invariant that satisfies $(dK/dx) = 0$, and Eqs. (4)–(6) clearly are the quantum analogue of Ermakov system of equations (1)–(3).

Recently, we have shown (Kaushal and Parashar, 1996) that such an ad-hoc identification of classical quantities x, t, and ω is not at all necessary to obtain the SE (4) and other structures (5) and (6); rather the SE (4) itself admits a class of solutions of the type (7), which in turn provides the space invariant (5) and the Milne equation (6) in a natural manner. Not only this, within this framework both Schrodinger QM and supersymmetric QM are shown to have their roots in what is known as Riccati equation.

3. A GENERAL RESULT AND ITS APPLICATIONS

Considering these developments as well as the merit and magnitude of applicability of these concepts, a general result in one dimension can be stated as follows.

Theorem 1. *If $\psi(x)$ satisfies the SE,*

$$\psi''(x) + k^2(x)\,\psi(x) = 0, \quad k^2(x) = 2m[E - V(x)]/\hbar^2, \tag{9}$$

and differs from a real function $\psi_0(x)$ by a phase factor, that is,

$$\psi(x) = \psi_0(x)\ \exp[i\ f(x)], \quad f(x) = C\int^x \frac{dx'}{\psi_0^2}, \tag{10}$$

then $\psi_0(x)$ satisfies the Milne equation (Kaushal and Parashar, 1996)

$$\psi_0''(x) + k^2(x)\,\psi_0(x) = C^2/\psi_0^3(x), \tag{11}$$

and there exists a functional invariant,

$$K = C^2(\psi/\psi_0)^2 + (\psi_0\psi' - \psi_0'\psi)^2, \tag{12}$$

with respect to the space variable x.

Proof: Using the ansatz (10) in (9), and then equating the real and imaginary parts in the resultant expression to zero, one obtains

$$\psi_0'' + [k^2(x) - f'^2]\,\psi_0 = 0,$$
$$\psi_0 f'' + 2f'\psi_0' = 0.$$

Here the integration of the second equation while immediately yields

$$f' = \frac{C}{\psi_0^2} \quad \text{or} \quad f(x) = C \int^x \frac{dx'}{\psi_0^2},$$

with constant of integration taken as zero, the use of this form of $f(x)$ in the first equation provides the Milne from (11). Further, the elimination of $k^2(x)$ from (9) and (11), and subsequently the integration of the resultant expression (after using the integrating factor $2(\psi_0'\psi - \psi_0\psi')$) yields the invariant (12). □

Also note that results (11) and (12) remain unchanged even if one uses the ansatz

$$\psi(x) = N\psi_0(x) \ \sin f(x),$$

instead of (10). In this case, however, one has to equate the coefficients of orthogonal sine and cosine functions to zero.

Recall that a similar prescription was used (Holland, 1993) in the alternative de Broglie–Bohm interpretation of the QM in which, using a transformation similar to (10), the SE is replaced by two equations; one turns out to be the Hamilton–Jacobi equation with the quantum potential and the other is the continuity equation for the probability. In the present case, however, the space invariant (11) and the Milne equation (12) are obtained. One can as well look for a possible link between the invariant (10) and the quantum potential of the de Broglie–Bohm theory (Holland, 1993).

As an application of Theorem 1, we discuss here a method that helps in accounting for the phase in the Wigner function approach to QM. As a matter of fact the Wigner function $W(x, p)$, defined as a particular type of phase space density, however, reduces to position or momentum probability densities after integrating over p or x variables, respectively. Somehow the role of phase in its use in physical problems was not transparent. In the spirit of Theorem 1, one can as well write the stationary state $\psi(x)$ satisfying the SE (9), in the form

$$\psi(x) = \sqrt{I_1(x)} \ \exp[i \ f_1(x)], \tag{13}$$

where $I_1(x)$, defined by $I_1(x) = \int W(x, p)\, dp$ (Manfredi *et al.*, 1993), is assumed to be positive. Using (13) one can immediately obtain the Ermakov-type space invariant as

$$\mathcal{K} = \left\{ C_1^2\psi^2(x) + \left[I_1(x)\psi'(x) - \frac{1}{2}\psi(x)I_1'(x) \right]^2 \right\} \Big/ I_1(x), \tag{14}$$

where the phase function $f_1(x)$ is given by

$$f_1(x) = C_1 \int [I_1(x')^{-1}]\, dx'.$$

Similarly, if one writes the momentum wave function $\tilde{\psi}(p)$, as

$$\tilde{\psi}(p) = \sqrt{I_2(p)}\ \exp[i\ f_2(p)], \tag{15}$$

where $\tilde{\psi}(p)$ satisfies the one-dimensional SE,

$$[d^2\psi^2(p)/dp^2] + q^2(p)\tilde{\psi}(p) = 0,$$

with $q^2(p) = [2(E - p^2/2m)/\chi\hbar^2]$, and $I_2(p)$, defined by $I_2(p) = \int W(x, p)\, dx$, is assumed to be positive, then an Ermakov-type invariant with respect to the momentum variable can also be written as

$$\tilde{\mathcal{K}} = \left\{C_2^2\tilde{\psi}^2(p) + \left[I_2(p)\tilde{\psi}'(p) - \frac{1}{2}\tilde{\psi}(p)I_2'(p)\right]^2\right\}\Big/ I_2(p), \tag{16}$$

where $f_2(p)$ is given by

$$f_2(p) = C_2 \int [I_2(p')]^{-1}\, dp'.$$

Here the primes on $\tilde{\psi}$ and I_i indicate differentiation with respect to the corresponding argument and the replacement $x \longrightarrow i\,\hbar(\partial/\partial p)$ is used in the potential $V(x) = (1/2)\chi x^2$. Unfortunately, such a position-momentum symmetry in the SE is restricted only to this harmonic oscillator potential, which again gives rise to the Gaussian form of the wave packet—a concept often used in illustrating the viability of Wigner functions in various phenomena. Although the utility of Wigner function approach to QM is limited mainly to the situations where the phase of the wavefunction is not important, instead $|\psi|^2$ is sufficient for this purpose, but the multidimensional Ermakov theory brings in the role of phase in this well-known approach and thereby enriches its domain of applications.

Another application of the above general result has been investigated (Kaushal, 1998) in the context of Tolman–Oppenheimer–Volkoff theory of stellar structure (Weinberg, 1972). Here, by converting the Riccati-type pressure equation into a Schrodinger-like equation, the existence of a space invariant is shown (Kaushal, 1998) in the theory. This invariant is supposed to provide a check on the variety of equations of state obtained from different models and used to study the stability of stellar objects in recent times. Also, one can as well use the present result to investigate the Wheeler–DeWitt equation with reference to the cosmological singularities (Kim, 1997; Weinberg, 1972) of the universe.

4. GENERALIZATION TO THREE DIMENSIONS AND APPLICATIONS

With a view to explore further applications of the general result of Section 3, a straightforward generalization to three dimensions can be carried out and the same can be stated (Kaushal, 1996) as follows.

Theorem 2. *If $\psi(\vec{x})$ satisfies the SE,*

$$[\nabla^2 + k^2(\vec{x})]\,\psi(\vec{x}) = 0, \quad k^2(\vec{x}) = 2m[E - V(\vec{x})]/\hbar^2, \tag{17}$$

and differs from $\psi_0(\vec{x})$ by a phase factor, viz.,

$$\psi(\vec{x}) = \psi_0(\vec{x})\ \exp[i\ f(\vec{x})], \quad f(\vec{x}) \equiv f = \int^x \frac{\vec{C}\cdot d\vec{x}'}{\psi_0^2}, \tag{18}$$

then $\psi_0(\vec{x})$ satisfies the Milne equation

$$[\nabla^2 + k^2(\vec{x})]\,\psi(\vec{x}) = \vec{C}^2/\psi_0^3(\vec{x}), \tag{19}$$

and correspondingly there exists an Ermakov-type functional invariant

$$K = \vec{C}^2\left(\frac{\psi}{\psi_0}\right)^2 + (\psi_0\vec{\nabla}\psi - \psi\vec{\nabla}\psi_0)^2, \tag{20}$$

with respect to the space variable $\vec{x}$. Here $\vec{C}$ is a vector constant.

Proof: As before, using ansatz (18) in (17) and subsequently equating the real and imaginary parts of the resultant expression separately to zero one obtains,

$$\nabla^2\psi_0 - (\nabla f)^2\psi_0 + k^2\psi_0 = 0, \tag{21a}$$

$$(\nabla^2 f)\psi_0 + 2\nabla f \cdot \vec{\nabla}\psi_0 = 0. \tag{21b}$$

Equation (21b), when expressed as $\nabla\cdot(\psi_0^2\nabla f) = 0$, implies $\psi_0^2\nabla f =$ constant (say $\vec{C}$). On the other hand, $\psi_0^2\nabla f$ can also be expressed as the curl of a vector function $\vec{G}$. Although the uniqueness of the solution of (21b) demands $\vec{\nabla}\times\vec{G} = \vec{C}$, a general choice could be

$$\psi_0^2\nabla f = \vec{C} + C_1(\nabla\times\vec{G}), \tag{22}$$

where C_1 is a scalar constant and we choose it to be zero for the moment. This leads to

$$\nabla f = \vec{C}/\psi_0^2 \quad \text{or} \quad f = \int^{(\vec{x})} \frac{\vec{C}\cdot d\vec{x}'}{\psi_0^2}. \tag{23}$$

Use of this form of ∇f in (21a) will immediately yield the Milne equation (19).

Now eliminate $k^2(\vec{x})$ from (17) and (19) and this will provide the expression

$$\psi_0^2\nabla^2\psi - \psi\nabla^2\psi_0 = -\vec{C}^2\left(\psi/\psi_0^3\right),$$

which, after using the integrating factor, yields the functional invariant (20) as before. □

An important theme of this theorem is that there exists a built-in phase of fundamental nature in the quantum wave function (QWF) and the same can be attributed to the forces that are not accounted for by the potential $V(\vec{x})$ in (17). Here the existence of the invariant K is basically the manifestation of this fundamental phase. This is not surprising. However, a situation similar to this, without recognizing the existence of K, is in fact, known in anyon field theory where the nonlocal interactions due to the exchange potential, not accounted for by $V(\vec{x})$, manifest through the phase in the wave function.

Next we demonstrate the existence of an Ermakov-type functional invariant for (a) NLSE and (b) SE for a charged particle moving in a magnetic field. The role of the derived space invariant in these cases is expected to manifest in the theoretical understanding of the concerned physical phenomenon. For example, the study of case (a) could be helpful with reference to the recent applications (Dixon *et al.*, 1992; Gagnon and Winternitz, 1988, 1989; Tuszynski and Dixon, 1989a,b) of the NLSE, and that of (b) can provide a better (and perhaps more plausible) understanding of the A-B effect.

(a) *Nonlinear Schrodinger Equation:* In three dimensions, the TD NLSE can be written as

$$i\,\hbar\frac{\partial\Psi}{\partial t} = \nu_0\Psi - \frac{1}{2}\nabla^2\Psi + F_0\Psi^+\Psi\,\Psi. \tag{24}$$

The TD part can be separated by performing the well-known transformation

$$\Psi(\vec{x},t) = \psi(\vec{x})e^{-\iota\nu_1 t/\hbar}. \tag{25}$$

This leads to an equation of the form

$$\nabla^2\psi(\vec{x}) + 2(\nu_1-\nu_0)\psi(\vec{x}) - 2F_0|\psi|^2\psi = 0. \tag{26}$$

For the solution of (26), now make an ansatz similar to (18). Although this immediately provides the space invariant (20) by eliminating the terms $(\nu_1-\nu_0)$, as before, it yields a nonlinear equation for ψ_0 as

$$\nabla^2\psi_0 + 2(\nu_1-\nu_0)\psi_0 - 2F_0\psi_0^3 = \vec{C}^2/\psi_0^3, \tag{27}$$

where C_1 in (22) is again assumed to be zero. Note that the differential equation (27) is more complicated than the Milne equation (19) in spite of the fact that it now involves only constant coefficients.

(b) *Aharonov–Bohm effect:* This well-known effect (Aharonov and Bohm, 1959; Bohm and Hiley, 1979), observed experimentally (Chambers, 1960) and now discussed in several textbooks (Ballentine, 1990; Ryder, 1986; Sakurai, 1984), deals mainly with a shift of (or some effect on) the interference pattern of two electron beams in the double-slit experiment in the presence of a nonvanishing magnetic field but localized only in a region that does not intersect with the path of the electron beams. In fact, the position of the fringes within diffraction pattern shift systematically as the magnetic flux is varied, but their intensities change simultaneously so that the centroid of the diffraction pattern does not move.

Note that there is no explanation of this effect in terms of the classical Lorentz force; it has been understood only at the qunatum level and that too in terms of the phase of the wavefunction appearing in the form of a line integral $\int \vec{A} \cdot \vec{d\ell}$, where $\vec{A}$ is the vector potential. In relation to a complete understanding of the experimental results, it seems that the appearance of this phase in the QWF explains only the shift of the interference fringes and for the variation of their intensity something more is required in the existing theoretical framework. An account of the electromagnetic angular momentum, on the lines of Peshkin (1981), can also be only a partial explanation for this purpose. In what follows, we assume that the QWF $\psi_0(\vec{x})$ attains a phase $f(\vec{x})$ and becomes $\psi(\vec{x})$ due to the presence of some spurious interactions that are not accounted for by the potential term in the SE. This may very well be due to limitations of the measuring instrument or else can be attributed to the nonlocal nature of the underlying interactions. In this case, one cannot expect any visible effect as such at the classical level since $|\psi(\vec{x})|^2 = |\psi_0(\vec{x})|^2$; however, at the quantum level there could be manifestations of such a phase in terms of an Ermakov-type functional invariant.

For a charged particle moving in a magnetic field, the Hamiltonian can be written as

$$H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + e\,A_0, \tag{28}$$

where $A_0(\vec{x})$ is the scalar potential and $V(\vec{x}) = eA_0(\vec{x})$, and the corresponding SE takes the form

$$\left(\nabla - \frac{ie}{\hbar c}\vec{A}\right)^2 \Psi(\vec{x}) + k^2(\vec{x})\Psi(\vec{x}) = 0. \tag{29}$$

For $\Psi(\vec{x})$ we again make an ansatz as before, that is,

$$\Psi(\vec{x}) = \psi_0\,(\vec{x})\ \exp[i\ f(x)], \tag{30}$$

but now $\psi_0(\vec{x})$ corresponds to the case when the field $\vec{B} = 0$ and $V(\vec{x})$ remains the same as $eA_0(\vec{x})$. Since

$$\left(\nabla - \frac{ie}{\hbar c}\vec{A}\right)\Psi(\vec{x}) = \left[\nabla\psi_0\ +\ i\left(\nabla f - \frac{e}{\hbar c}\vec{A}\right)\psi_0\right]\ \exp(if), \tag{31}$$

the requirement that the operator $(\nabla - (ie/\hbar c)\vec{A})$ while operating on $\Psi(\vec{x})$ leaves the phase factor intact—a condition necessary for preserving the gauge invariance at the quantum level—will demand that the imaginary part in the bracket on the right-hand side (RHS) of (31) must identically vanish, leading to, $\nabla f = (e/\hbar c)\vec{A}$, which implies

$$f(\vec{x}) = \frac{e}{\hbar c}\int \vec{A}(\vec{x}')\cdot d\vec{x}'. \tag{32}$$

Interestingly, this is the standard form of the phase usually discussed (Aharonov and Bohm, 1959; Ballentine, 1990; Bohm and Hiley, 1979; Chambers, 1960; Peshkin, 1981; Ryder, 1986; Sakurai, 1984) in the context of A-B effect and also connected with the gauge transformation. Let us see what happens when $[\nabla - (ie/\hbar c)\vec{A}]^2$ operates on $\Psi(\vec{x})$ [this is what is actually needed in (29)]. In other words, the use of (30) in (29) after separating the real and imaginary parts implies

$$\nabla^2\psi_0 + \left[-(\nabla f)^2\psi_0 + \frac{e}{\hbar c}\nabla f \cdot \vec{A}\psi_0 + \frac{e}{\hbar c}(\vec{A}\cdot\nabla f)\psi_0 - \frac{e^2\vec{A}^2}{\hbar^2 c^2}\psi_0 + k^2\psi_0 \right] = 0, \tag{33a}$$

$$\nabla^2 f \cdot \psi_0 + 2\nabla f \cdot \nabla\psi_0 - \frac{2e}{\hbar c}\vec{A}\cdot\nabla\psi_0 - \frac{e}{\hbar c}\nabla\cdot\vec{A}\psi_0 = 0. \tag{33b}$$

Equation (33b), in general, immediately yields

$$\nabla f = \frac{1}{\psi_0^2}(\vec{C} + C_1\nabla\times\vec{G}) + \frac{e}{\hbar c}\vec{A} \tag{34a}$$

or

$$f = \int \left[\frac{(\vec{C} + C_1\nabla\times\vec{G})}{\psi_0^2} + \frac{e}{\hbar c}\vec{A}(\vec{x}') \right] \cdot d\vec{x}', \tag{34b}$$

and with this form of ∇f, Eq. (33a) reduces to the Milne equation, same as (19) but for the $\psi_0(\vec{x})$ defined now in ansatz (30). However, the functional form appearing on the RHS of (20) is no more a space invariant; it is now given by

$$\nabla\left[\vec{C}^2\left(\frac{\Psi}{\psi_0}\right)^2 + (\psi_0\nabla\Psi - \Psi\nabla\psi_0)^2\right] = \frac{-2ie}{\hbar c}\psi_0^3\nabla\left(\frac{\Psi}{\psi_0}\right) \times \left[\nabla\cdot(\vec{A}\Psi) - \frac{ie}{\hbar c}\vec{A}^2\Psi\right], \tag{35}$$

where C_1, for simplicity, is chosen to be zero. It may be mentioned that while the second term in the phase (34b) is in accordance with the gauge invariance of H in (28) and also it is in conformity with the kinetic term in it, in the form (32) it has very widely been used (Ballentine, 1990; Ryder, 1986; Sakurai, 1984) to find an explanation of the A-B effect. However, the appearance of the first term in (34b) is the consequence of Theorem 2 and has not been discussed earlier in the literature to the best of my knowledge.

The fact that the RHS of (35) in the present case turns out to be nonzero, can be attributed to the occurence of the A-B effect. Alternatively, Eq. (35) can be written in terms of another functional invariant, K, as

$$K = \mathcal{K} + \frac{2ie}{\hbar c} \int^{\vec{x}} \psi_0^3(\vec{x}') \left[\nabla' \cdot (\vec{A}\Psi) - \frac{ie}{\hbar c} \vec{A}^2 \Psi \right] \nabla' \left(\frac{\Psi}{\psi_0} \right) d\vec{x}', \tag{35'}$$

where

$$\mathcal{K} = \vec{C}^2 \left(\frac{\Psi}{\psi_0} \right)^2 + (\psi_0 \nabla \Psi - \Psi \nabla \psi_0)^2, \tag{36}$$

and note that K reduces to $\mathcal{K}$ for $\vec{A} = 0$.

Clearly, the condition for the nonvanishing of the second term on the RHS of (35′) is not only $\vec{A}(\vec{x}') \neq 0$ but also $\vec{A}(x')$ should be in conformity with

$$\nabla' \cdot (\vec{A}\Psi) \neq \frac{ie}{\hbar c} \vec{A} \cdot (\vec{A}\Psi), \tag{37}$$

that is, if we define a matter-field-coupling current $\vec{j} = \vec{A}\Psi$, then the divergence of this current should overbalance the self-interaction or the source term $\vec{j} \cdot \vec{A}$. In other words, it is the matter-field-coupling current of nonlocal nature, which comes into play in the presence of $\vec{B}$ and is the cause of the A-B effect. Consider the situation when $\vec{\nabla}' \cdot \vec{j} = (ie/\hbar c)\, \vec{A} \cdot \vec{j}$, but $\vec{A} \neq 0$. In that case there will be A-B effect mainly due to the presence of Ψ (in place of ψ) in (36). However, if $\vec{A} = 0$, there will not be any A-B effect. Thus, in addition to the necessary condition, $\vec{A} \neq 0$, for the occurence of the A-B effect, the condition (37) is a sufficient condition, which perhaps can be useful in explaining the intensity distribution of the interference fringes.

Now the question arises whether we can have a nonlocal vector potential corresponding to a local field $\vec{B}$. In view of Theorem 2, the answer is yes. In fact, the applied $\vec{B}$ in the double-slit experiment leading to the flux $\Phi = \int \vec{B} \cdot d\vec{S}$, corresponds to whole of the phase (34b), which after using the Stokes theorem, would imply

$$\vec{B} = \vec{\nabla} \times \vec{A}_{\mathrm{eff}} \tag{38a}$$

where

$$\vec{A}_{\mathrm{eff}} = \frac{\vec{C}}{\psi_0^2} + \frac{e}{\hbar c} \vec{A}. \tag{38b}$$

It is interesting to note that the first term in (38b) can give rise to the nonlocal contribution, whereas the second term contributes only locally due to the presence of the same $\vec{B}$. This is mainly because the ∇-operator and the argument of ψ_0 in the first term of (38) belong to different but related coordinate systems, which, however, is not the case with the second term. Moreover, the appearance of the term $(\vec{\nabla} \times \vec{G})$ in (38b) [without choosing C_1 as zero in (22)] will further bring in the nonlocal character.

Finally, a remark about the quantization condition is worth mentioning. Following the earlier works (Kaushal and Parashar, 1996; Korsch and Laurent, 1981; Korsch *et al.*, 1982; Lee, 1982, 1984), now it is not difficult to derive a new quantization rule in the present context. In fact, the single-valued nature of the QWF would demand that the phase f [cf. Eq. (34b)] has to be an integral multiple of 2π and thus leads (assuming $C_1 = 0$) to

$$\int_{-\infty}^{\infty} \left(\frac{C_x}{\psi_0^2} + \frac{e}{\hbar c} A_x \right) dx = (n+1)\pi, \quad n = 0, 1, 2, \ldots, \tag{39}$$

along each degree of freedom. This is some sort of a modified version of the Milne quantization condition (Korsch and Laurent, 1981) in the presence of magnetic field. However, the flux quantization (see, e.g., Sakurai, 1984) condition for the field $\vec{B}$ defined through (38) remains intact.

5. CONCLUDING DISCUSSION

A study of the quantum analogue of the classical mechanics of time-dependent harmonic oscillator (called multidimensional Ermakov theory in the literature) has led to a general mathematical result established here in the form of a theorem. Although only a few applications and implications of this theorem are demonstrated here in one and three dimensions, several others are worth investigating. In brief, the scope of applications of this theorem, in general, can be stated as follows: If the theoretical understanding of a physical phenomenon involves the Schrodinger, nonlinear Schrodinger, Riccati or Milne (or the likes) equation, a space invariant exists for the system whose role with reference to the measurable quantities in that phenomenon vis-a-vis the geometry of the system cannot be ruled out. As far as the solution of the corresponding underlying nonlinear differential equation is concerned, the same can either be handled analytically or numerically without affecting the physical content of the problem. Moreover, the existence of such a space invariant appears to be a common feature of coupled differential equations, particularly the ones investigated (Roy, 1988) in the context of one-dimensional Dirac equation or the ones which appear in the theory (Yurke and Stoler, 1995) of optical experiments performed to test the Bell inequality/EPR paradox.

The theorem suggests the appearance of a fundamental phase f in the QWF, which can be attributed to some unaccounted nonlocal interactions and manifests through the invariant K with respect to the space variable. Following Eliezer and Gray (1976), if one accepts the interpretation of K as the angular momentum in a projected two-dimensional plane (which, of course, in the present case is a function space), then the errors in the measurements of K and f (denoted by ΔK and Δf, respectively) can be assumed to be related through the Heisenber-type uncertainty relation,

$$\Delta K \cdot \Delta f \sim \hbar. \tag{40}$$

This, of course, sets the limit on the simultaneous measurements of K and f. Although we do not derive relation (40) as such here, but it may be mentioned that in the case of quantized radiation field the observables corresponding to the phase ($\hat{\phi}$) and number ($\hat{N}$) operators satisfy (see, e.g., Sakurai, 1984) an uncertainty relation of the type $\Delta\phi \cdot \Delta N \geq 1$. Interestingly, what relation (40) represents is the analogue to this result in the Schrodinger QM. Alternatively, K can also be interpreted (Kaushal and Parashar, 1996) as an invariant-energy-functional in the function space of ψ and ψ_0.

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